

Isomorphism Theorems for Semigroups of Order-preserving Full Transformations

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Abstract : For each subchain \mathcal{C}' of a chain \mathcal{C} , the semigroup of all full order-preserving transformations, $\alpha : \mathcal{C} \rightarrow \mathcal{C}'$ satisfying if $x \leq y$ implies $x\alpha \leq y\alpha$ for $x, y \in \mathcal{C}$, is denoted by $T_{OP}(\mathcal{C}, \mathcal{C}')$. It is well-known that for any posets X and Y , $T_{OP}(X) \cong T_{OP}(Y)$ if and only if X and Y are either order-isomorphic or order-anti-isomorphic. The purpose of this paper is to show the analogous results for $T_{OP}(X, X')$ and $T_{OP}(Y, Y')$.

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1 Introduction and Preliminaries

For a nonempty set X , let $T(X)$ be the full transformation semigroup under composition of all maps from X to X . When X is a partially ordered set (poset), a mapping α in $T(X)$ is called *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$ for $x, y \in X$, and α is called *regressive* (or *order-decreasing*) if $x\alpha \leq x$ for all $x \in X$. We denote the set of all order-preserving maps by $T_{OP}(X)$ and denote the set of all regressive maps by $T_{RE}(X)$.

The order-preserving transformation semigroup was first introduced by Howie [1]. It is known that for any posets X and Y , $T_{OP}(X) \cong T_{OP}(Y)$ if and only if X and Y are either order-isomorphic or order-anti-isomorphic (see [2], page 222 and 223). In 1992, Umar [7] proved that for any chains X and Y , $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic. Later, T. Saito, et al. [3] generalized this result on partially ordered sets.

In 1975, J. S. V. Symons [4] introduced the subsemigroup $T(X, X')$ consisting of $\alpha \in T(X)$ with $\text{ran } \alpha \subseteq X' \subseteq X$. This motivated us to study $T_{RE}(X, X')$, a subsemigroup of full regressive transformations and in particular $T_{RE}(X) = T_{RE}(X, X)$. To classify the subsemigroups of all regressive transformations on chains, in [5, 6], we restructure a chain \mathcal{C} based on a fixed subchain \mathcal{C}' of \mathcal{C} . By considering \mathcal{C}' as the skeleton of \mathcal{C} and grouping elements

in $\mathcal{C} \setminus \mathcal{C}'$ into classes, we have that each class contains all elements in $\mathcal{C} \setminus \mathcal{C}'$ which has no elements in \mathcal{C}' lie between them. We denote this class as follows:

$$[\mathcal{C} \setminus \mathcal{C}'] = \{ [k] : k \in \mathcal{C} \setminus \mathcal{C}' \text{ and } k > c_1 > c_2 \text{ for some } c_1, c_2 \in \mathcal{C}' \}.$$

Then $\mathcal{C}' \cup [\mathcal{C} \setminus \mathcal{C}']$ becomes a chain under the partial order induced by the chain \mathcal{C} in the natural way. Denote this chain by $\mathcal{A}(\mathcal{C}, \mathcal{C}')$ and call it the *adjusted chain* of \mathcal{C} with respect to \mathcal{C}' .

For subchains X' and Y' of chains X and Y , respectively, $\mathcal{A}(X, X')$ is *order-(anti)-structural isomorphic* to $\mathcal{A}(Y, Y')$ if there is an order-(anti)-isomorphism $\zeta : \mathcal{A}(X, X') \rightarrow \mathcal{A}(Y, Y')$ satisfying the following conditions:

1. the restriction of ζ on X' is onto Y' ,
2. for each $[k] \in \{X \setminus X'\}$, $[k]$ and $\zeta([k])$ have the same cardinality.

We proved that $T_{RE}(X, X') \cong T_{RE}(Y, Y')$ if and only if $\mathcal{A}(X, X')$ is order-structural isomorphic to $\mathcal{A}(Y, Y')$ (see [6]).

It is natural to ask whether the above result holds or not for order-preserving transformation semigroups.

Example. Although two chains $[0, 1]$ and $[0, 1] \cup \{2\}$ are not order-(anti)-isomorphic, $T_{OP}([0, 1], \{0, 1\})$ and $T_{OP}([0, 1] \cup \{2\}, \{0, 2\})$ are isomorphic. To see this, observe that there are only two right zero elements whereas there are uncountable infinite right identity elements in each $T_{OP}([0, 1], \{0, 1\})$ and $T_{OP}([0, 1] \cup \{2\}, \{0, 2\})$.

The purpose of this paper is to show that when $|X'| \geq 5$ and $T_{OP}(X, X') \cong T_{OP}(Y, Y')$, there is an order-(anti)-isomorphism f from X onto Y such that $f(X') = Y'$. The techniques of some parts of our proof are obtained from [6].

For convenience, we give some definitions and notations here: Let \mathcal{C}' be a subchain of a chain \mathcal{C} .

- Set $\mathcal{A}\{\mathcal{C}, \mathcal{C}'\}$ as the adjusted chain of \mathcal{C} with respect to \mathcal{C}' such that $\{\mathcal{C} \setminus \mathcal{C}'\}$ is an equivalence class of $\mathcal{C} \setminus \mathcal{C}'$.
- For $a, b \in \mathcal{C}$ with $a < b$ and $[c] \in \{\mathcal{C} \setminus \mathcal{C}'\}$, we define

$$\begin{aligned} [a, b] &= \{z \in \mathcal{C} : a \leq z \leq b\}, & (a, b) &= \{z \in \mathcal{C} : a < z < b\}, \\ (a, b] &= \{z \in \mathcal{C} : a < z \leq b\}, & [a, b) &= \{z \in \mathcal{C} : a \leq z < b\}, \\ (\leftarrow a] &= \{z \in \mathcal{C} : z \leq a\}, & [a \rightarrow) &= \{z \in \mathcal{C} : z \geq a\}, \\ (\leftarrow a) &= \{z \in \mathcal{C} : z < a\}, & (a \rightarrow) &= \{z \in \mathcal{C} : z > a\}, \\ (\leftarrow [c]) &= \{z \in \mathcal{C} : z < [c]\}, & ([c] \rightarrow) &= \{z \in \mathcal{C} : z > [c]\}. \end{aligned}$$

- A nonempty subset C of a chain \mathcal{C} is called *convex* if $x, y \in C$ and $x \leq z \leq y \Rightarrow z \in C$. A convex subset C of \mathcal{C} is called a *top convex* of \mathcal{C} if there is no upper bound of C in \mathcal{C} and is called a *bottom convex* of \mathcal{C} if there is no lower bound of C in \mathcal{C} .
- For $\alpha \in T(\mathcal{C})$, $F(\alpha) = \{x \in \mathcal{C} : x\alpha = x\}$.

2 Preserving skeletons

Given an order-preserving map $\alpha : X \rightarrow X'$, the skeleton of α consists of the partial map of α by restricted its domain on X' and $\text{ran } \alpha$, such as the following map:

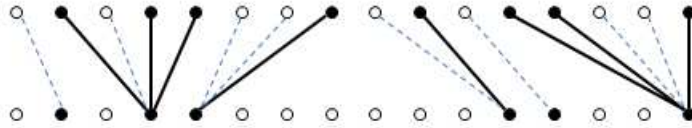


Figure 1: The skeleton of an order-preserving map

Theorem 2.1. *If $T_{OP}(X, X') \cong T_{OP}(Y, Y')$, then X' and Y' are either order-isomorphic or order-anti-isomorphic.*

Proof. Let $\varphi : T_{OP}(X, X') \rightarrow T_{OP}(Y, Y')$ be an isomorphism. For each $a \in X'$, there is an element $\bar{a} \in Y'$ such that $(X_a)\varphi = Y_{\bar{a}}$ by idempotent and right zero properties of X_a and $Y_{\bar{a}}$. The map $a \mapsto \bar{a}$ becomes an bijective map from X' onto Y' . It remains to show that this map is either order-preserving or order-reversing. Let $a, b, s, t \in X'$ be such that $a < b$ and $s < t$. Then $\begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} \in T_{OP}(X, X')$ such that

$$X_a \begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} = X_s \text{ and } X_b \begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} = X_t.$$

Then

$$Y_{\bar{a}} \left(\begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} \varphi \right) = Y_{\bar{s}} \text{ and } Y_{\bar{b}} \left(\begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} \varphi \right) = Y_{\bar{t}}.$$

Consequently,

$$\bar{a} \begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} \varphi = \bar{s} \text{ and } \bar{b} \begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} \varphi = \bar{t}.$$

Since $\begin{pmatrix} (\leftarrow a] & (a \rightarrow) \\ s & t \end{pmatrix} \varphi \in T_{OP}(Y, Y')$, it follows that if $\bar{a} < \bar{b}$ then $\bar{s} < \bar{t}$ and if $\bar{a} > \bar{b}$ then $\bar{s} > \bar{t}$. This proves that X' and Y' are either order-isomorphic or order-anti-isomorphic. \square

From now until the end of this paper, we let $\varphi : T_{OP}(X, X') \rightarrow T_{OP}(Y, Y')$ be an isomorphism. The order-(anti)-isomorphism from X' onto Y' , defined in the proof of Theorem 2.1, will be denoted by θ_φ . It is easily seen that the order-(anti)-isomorphism $\theta_{\varphi^{-1}}$ from Y' onto X' , induced by the isomorphism φ^{-1} , is the inverse function of θ_φ . That is,

$$\theta_{\varphi^{-1}} = (\theta_\varphi)^{-1}.$$

To study the structure of α and $\alpha\varphi$ through θ_φ when $\alpha \in T_{OP}(X, X')$, we first prove the following lemmas.

Lemma 2.2. *For each $\alpha \in T_{OP}(X, X')$, the following statements hold:*

- (i) $(F(\alpha))\theta_\varphi = F(\alpha\varphi)$.
- (ii) For $a \in \text{ran } \alpha$ such that $a\alpha^{-1} \cap X' \neq \emptyset$,
 $\bar{a} \in \text{ran } (\alpha\varphi)$ and $\bar{a}(\alpha\varphi)^{-1} \cap Y' = (a\alpha^{-1} \cap X')\theta_\varphi$.

In particular, if α is an idempotent, then $(\text{ran } \alpha)\theta_\varphi = \text{ran } (\alpha\varphi)$.

Proof. (i) Let $a \in F(\alpha)$. Then $a\alpha = a$. Since $Y_{\bar{a}}(\alpha\varphi) = (X_a\varphi)(\alpha\varphi) = (X_a\alpha)\varphi = X_a\varphi = Y_{\bar{a}}$, it follows that $\bar{a}(\alpha\varphi) = \bar{a} = a\theta_\varphi \in F(\alpha\varphi)$. Similarly, if $\bar{s} \in F(\alpha\varphi)$, then $X_s\alpha = (Y_{\bar{s}}\varphi^{-1})\alpha = (Y_{\bar{s}}(\alpha\varphi))\varphi^{-1} = (Y_{\bar{s}})\varphi^{-1} = X_s$, that is, $s\alpha = s$. Then $\bar{s} = s\theta_\varphi \in (\text{ran } \alpha)\theta_\varphi$.

(ii) For $a \in \text{ran } \alpha$ such that $a\alpha^{-1} \cap X' \neq \emptyset$, let $x \in a\alpha^{-1} \cap X'$. Then $a \in F(X_x\alpha)$, by (i), $\bar{a} \in F((X_x\varphi)(\alpha\varphi))$. That is, $\bar{a} \in \text{ran } (\alpha\varphi)$. Since $\bar{x}(\alpha\varphi) = \bar{a}(X_x\varphi)(\alpha\varphi) = \bar{a}$, it follows that $\bar{x} \in \bar{a}(\alpha\varphi)^{-1} \cap Y'$. Then $(a\alpha^{-1} \cap X')\theta_\varphi \subseteq \bar{a}(\alpha\varphi)^{-1} \cap Y'$. Similarly, by considering φ^{-1} instead of φ , $\bar{a}(\alpha\varphi)^{-1} \cap Y' \subseteq (a\alpha^{-1} \cap X')\theta_\varphi$. Then we obtain the equality as required. \square

Lemma 2.3. *For each $\alpha \in T_{OP}(X, X')$, if $b \in \text{ran } \alpha$ and $b\alpha^{-1} \cap X' = \emptyset$, then $\bar{b} \in \text{ran } (\alpha\varphi)$.*

Proof. Suppose $b \in \text{ran } \alpha$ and $b\alpha^{-1} \cap X' = \emptyset$. Then $|X'| > 1$. Suppose b is neither maximum nor minimum in X' . We choose $a, c \in X'$ such that $a < b < c$ and set $\epsilon_b = \begin{pmatrix} \leftarrow b & b & (b \rightarrow) \\ a & b & c \end{pmatrix}$. Then ϵ_b is an idempotent with $b(\epsilon_b)^{-1} \cap X' = \{b\}$. By Lemma 2.2, $\bar{b}(\epsilon_b\varphi)^{-1} \cap Y' = \{\bar{b}\}$. Suppose in the contrary that $\bar{b} \notin \text{ran } (\alpha\varphi)$. Then we have $\bar{b} \notin \text{ran } ((\alpha\varphi)(\epsilon_b\varphi))$. Since $|\text{ran } (\alpha\varphi)(\epsilon_b\varphi)|$ is finite, this guarantees the existence of an idempotent μ in $T_{OP}(Y, Y')$ with $\text{ran } \mu = \text{ran } ((\alpha\varphi)(\epsilon_b\varphi))$. Then $\mu\varphi^{-1}$ is an idempotent in $T_{OP}(X, X')$, by Lemma 2.2, $b \notin \text{ran } (\mu\varphi^{-1})$. However

$$\alpha\epsilon_b(\mu\varphi^{-1}) = ((\alpha\varphi)(\epsilon_b\varphi)\mu)\varphi^{-1} = ((\alpha\varphi)(\epsilon_b\varphi))\varphi^{-1} = \alpha\epsilon_b,$$

a contradiction. For the case b is either maximum or minimum, we prove in the same way by letting ϵ_b as before and setting $a = b$ if b is minimum, and setting $c = b$ if b is maximum. \square

From Lemma 2.2 and 2.3, the following fact is directly obtained.

Proposition 2.4. *For each $\alpha \in T_{OP}(X, X')$, $(\text{ran } \alpha)\theta_\varphi = \text{ran } (\alpha\varphi)$.*

3 Gaps in skeletons

In this section, we first show that $\mathcal{A}\{X, X'\}$ and $\mathcal{A}\{Y, Y'\}$ are order-(anti)-structural isomorphic. Next, we prove that being order-(anti)-isomorphic between the corresponding gaps is a necessary condition.

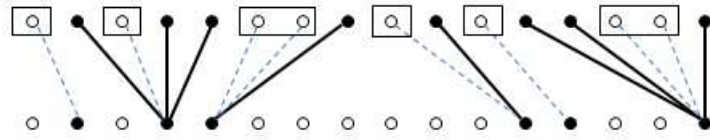


Figure 2: Gaps in the skeleton of an order-preserving map

To obtain the result, the following lemmas is required.

Lemma 3.1. *For $[k] \in \{X \setminus X'\}$ such that either $a < [k] < b < c$ or $a < b < [k] < c$ for some $a, b, c \in X'$, we have for each convex subset A of $[k]$,*

$$\begin{pmatrix} (\leftarrow A) & A & (A \rightarrow) \\ a & b & c \end{pmatrix} \mapsto \begin{pmatrix} (\leftarrow B) & B & (B \rightarrow) \\ \bar{a} & \bar{b} & \bar{c} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (\leftarrow B) & B & (B \rightarrow) \\ \bar{c} & \bar{b} & \bar{a} \end{pmatrix},$$

up to θ_φ , for some a convex subset B of a unique $[t_k] \in \{Y \setminus Y'\}$.

Proof. Set $\alpha = \begin{pmatrix} (\leftarrow A) & A & (A \rightarrow) \\ a & b & c \end{pmatrix}$. It is clear that α is in $T_{OP}(X, X')$. From $\text{ran } \alpha = \{a, b, c\}$, by Proposition 2.4, $\text{ran } (\alpha\varphi) = \{\bar{a}, \bar{b}, \bar{c}\}$. Since $(a\alpha^{-1} \cup c\alpha^{-1}) \cap X' = X'$, by Lemma 2.2, it follows that $(\bar{a}(\alpha\varphi)^{-1} \cup \bar{c}(\alpha\varphi)^{-1}) \cap Y' = Y'$. Since $\alpha\varphi$ is an order-preserving transformation on chain, there is a unique $[t_k] \in \{Y \setminus Y'\}$ having $\bar{b}(\alpha\varphi)^{-1}$ as a convex subset. \square

The following corollary is a direct consequence of Lemma 3.1.

Corollary 3.2. *For each $\alpha \in T_{OP}(X, X')$ and $b \in \text{ran } \alpha$ with $a < b < c$ for some $a, c \in X'$, the following statement holds:*

$b\alpha^{-1}$ is a convex subset of $[k]$ if and only if $\bar{b}(\alpha\varphi)^{-1}$ is a convex subset of $[t_k]$.

The following lemma is obtained directly by the same argument as in the proof of Lemma 3.1.

Lemma 3.3. *Suppose $[k]$ is maximum (or minimum) in $\mathcal{A}\{X, X'\}$ and $|X'| \geq 2$. Let $a, b \in X'$ be such that $a < b$, and a top convex subset A of $[k]$. Then*

$$\begin{pmatrix} (\leftarrow A) & A \\ a & b \end{pmatrix} \mapsto \begin{pmatrix} (\leftarrow B) & B \\ \bar{a} & \bar{b} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B & (B \rightarrow) \\ \bar{b} & \bar{a} \end{pmatrix},$$

up to θ_φ , for some a top (or bottom) convex subset B of the maximum (or minimum) $[t_k] \in \{Y \setminus Y'\}$.

Lemma 3.1 and 3.3 give a remarkable result as follows:

Proposition 3.4. *Let θ_φ be an order-(anti)-isomorphism from X' onto Y' , induced by an isomorphism $\varphi : T_{OP}(X, X') \rightarrow T_{OP}(Y, Y')$. Suppose that $|X'| = |Y'| > 1$. Then for each $[k] \in \{X \setminus X'\}$, there is a corresponding $[t_k] \in \{Y \setminus Y'\}$ such that the extended map of θ_φ from $X' \cup \{[k]\}$ onto $Y' \cup \{[t_k]\}$ is order-(anti)-isomorphic.*

Proof. Let $[k] \in \{X \setminus X'\}$. Suppose that $[k]$ is neither maximum nor minimum in $\mathcal{A}\{X, X'\}$. If there are $a, b, c \in X'$ such that either $a < [k] < b < c$ or $a < b < [k] < c$, then by an order-preserving transformation defined in Lemma 3.1, there is the corresponding gap $[t_k]$. Suppose there are only two elements, namely a and b , in X' such that $[k]$ lies between them. We consider

$$S = \{ \alpha \in E(T_{OP}(X, X')) : \text{ran } \alpha = \{a, b\} \}.$$

Then by Proposition 2.4, we have

$$(S)\varphi = \{ \beta \in E(T_{OP}(Y, Y')) : \text{ran } \beta = \{\bar{a}, \bar{b}\} \}.$$

It is clear that $|S|$ depends on $|[k]|$ and $|S| = |(S)\varphi|$. These imply the existence of $[t_k] \in \{Y \setminus Y'\}$ such that $[t_k]$ lies between \bar{a} and \bar{b} .

By Lemma 3.3, the other cases is proved. □

From above proposition, the union of all these extensions forms an order-(anti)-isomorphism from $\mathcal{A}\{X, X'\}$ onto $\mathcal{A}\{Y, Y'\}$ sending $[k] \mapsto [t_k]$. Finally, we obtain the following result.

Corollary 3.5. *There is an order-(anti)-isomorphism from $\mathcal{A}\{X, X'\}$ onto $\mathcal{A}\{Y, Y'\}$ such that the restriction on X' is onto Y' .*

We have immediately the following result for the special case.

Proposition 3.6. Suppose $|X'| \geq 2$. If $[k] \in \{X \setminus X'\}$ such that $||[k]|| < \infty$, then $||[k]|| = ||[t_k]||$.

In general, to show that the corresponding gaps $[k]$ and $[t_k]$ have the same cardinality, we work on the assumption $|X'| \geq 4$, and prove the following lemma:

Lemma 3.7. For each $x \in [k]$ and $a_1 < a_2 < a_3 < a_4$ in X' , there is $t \in [t_k]$ such that for $i = 2$ or 3 ,

$$\begin{pmatrix} (\leftarrow x) & x & (x \rightarrow) \\ a_1 & a_i & a_4 \end{pmatrix} \mapsto \begin{pmatrix} (\leftarrow t) & t & (t \rightarrow) \\ \bar{a}_1 & \bar{a}_i & \bar{a}_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (\leftarrow t) & t & (t \rightarrow) \\ \bar{a}_4 & \bar{a}_i & \bar{a}_1 \end{pmatrix},$$

up to θ_φ .

Proof. Suppose that θ_φ is order-isomorphic and choose $i = 3$. We set $\alpha_x = \begin{pmatrix} (\leftarrow x) & x & (x \rightarrow) \\ a_1 & a_3 & a_4 \end{pmatrix}$. By Corollary 3.2, we obtain that $\bar{a}_3(\alpha_x \varphi)^{-1}$ is a convex subset of $[t_k]$. Suppose that $|\bar{a}_3(\alpha_x \varphi)^{-1}| > 1$. Then we can divide $\bar{a}_3(\alpha_x \varphi)^{-1}$ into a bottom and a top convex subsets of $\bar{a}_3(\alpha_x \varphi)^{-1}$, namely K_1 and K_2 , respectively. Define two order-preserving transformations in $T_{OP}(Y, Y')$ as follows:

$$\beta = \begin{pmatrix} K_1 & K_2 & y \\ \bar{a}_2 & \bar{a}_3 & y(\alpha_x \varphi) \end{pmatrix}_{y \notin K_1 \cup K_2} \quad \text{and} \quad \beta' = \begin{pmatrix} (\leftarrow \bar{a}_2) & [\bar{a}_2, \bar{a}_3] & (\bar{a}_3 \rightarrow) \\ \bar{a}_1 & \bar{a}_3 & \bar{a}_4 \end{pmatrix}.$$

It is clear that $\beta\beta' = \alpha_x \varphi$. By Corollary 3.2, $a_2(\beta\varphi^{-1})^{-1}$ and $a_3(\beta\varphi^{-1})^{-1}$ are two convex subsets of $[k]$. From $(\beta\varphi^{-1})(\beta'\varphi^{-1}) = \alpha_x$, it follows that $(a_2(\beta\varphi^{-1})^{-1} \cup a_3(\beta\varphi^{-1})^{-1})\alpha_x = \{a_3\}$, that is, $|a_3\alpha_x^{-1}| > 1$, a contradiction.

Similarly, for the case $i = 2$, therefore this lemma is proved. \square

Corollary 3.8. Suppose $|X'| \geq 4$. Then for each $[k] \in \{X \setminus X'\}$, $||[k]|| = ||[t_k]||$.

Proof. Since $||[k]|| = |\{ \{(\leftarrow x), x, (x \rightarrow)\} : x \in [k] \}|$ and by Lemma 3.7, these imply that $[k]$ and $[t_k]$ have the same cardinality. \square

Now, we have that when $|X'| \geq 4$, $\mathcal{A}\{X, X'\}$ and $\mathcal{A}\{Y, Y'\}$ are order-(anti)-structural isomorphic. To show that the corresponding gaps $[k]$ and $[t_k]$ are order-(anti)-isomorphic, we will work on the assumption $|X'| \geq 5$ and prove the following lemmas:

Lemma 3.9. For each $[k] \in \{X \setminus X'\}$, if $[k] = c\alpha^{-1}$ for some $\alpha \in T_{OP}(X, X')$ and $c \in \text{ran } \alpha$ with $a < b < c < d < e$ for some $a, b, d, e \in X'$, then $[t_k] = \bar{c}(\alpha\varphi)^{-1}$.

Proof. We assume that θ_φ is order-isomorphic. Define an idempotent $\epsilon_c = \begin{pmatrix} (\leftarrow c) & c & (c \rightarrow) \\ a & c & e \end{pmatrix}$. It is clear that $c(\alpha\epsilon_c)^{-1} = [k]$. By Lemma 3.1, we obtain that $\bar{c}((\alpha\varphi)(\epsilon_c\varphi))^{-1}$ is a convex subset of $[t_k]$. Suppose $\bar{c}((\alpha\varphi)(\epsilon_c\varphi))^{-1} \neq [t_k]$. WLOG, we will write $[t_k]$ as

$$B \dot{\cup} \bar{c}((\alpha\varphi)(\epsilon_c\varphi))^{-1} \dot{\cup} T$$

where B and T are the bottom and the top convex subsets of $[t_k]$, respectively. We remark here: If $[k]$ is minimum(or maximum) in $\mathcal{A}\{X, X'\}$, then $\text{ran}(\alpha\epsilon_c) = \{c, e\}$ (or $\text{ran}(\alpha\epsilon_c) = \{a, c\}$) and $B = \emptyset$ (or $T = \emptyset$). Then either $B \neq \emptyset$ or $T \neq \emptyset$. Assume that $B \neq \emptyset$. Then $B(\alpha\varphi)(\epsilon_c\varphi) = \{\bar{a}\}$, that is, $[k]$ is not minimum in $\mathcal{A}\{X, X'\}$. Then $\{\bar{a}, \bar{c}\} \subseteq \text{ran}((\alpha\varphi)(\epsilon_c\varphi)) \subseteq \{\bar{a}, \bar{c}, \bar{e}\}$. Define two order-preserving transformations in $T_{OP}(Y, Y')$ as follows:

$$\beta = \begin{pmatrix} B & y \\ \bar{b} & y(\alpha\varphi)(\epsilon_c\varphi) \end{pmatrix}_{y \notin B} \quad \text{and} \quad \beta' = \begin{pmatrix} (\leftarrow \bar{b}) & (\bar{b}, \bar{c}) & (\bar{c} \rightarrow) \\ \bar{a} & \bar{c} & \bar{e} \end{pmatrix}.$$

It is clear that $\beta\beta' = (\alpha\varphi)(\epsilon_c\varphi)$. By Proposition 2.4 and Lemma 3.1, we obtain that $b \in \text{ran}(\beta\varphi^{-1})$ and $b(\beta\varphi^{-1})^{-1}$ is a convex subset of $[k]$. Let $x \in b(\beta\varphi^{-1})^{-1}$. Then, by Lemma 2.2,

$$a = b(\beta'\varphi^{-1}) = x(\beta\varphi^{-1})(\beta'\varphi^{-1}) = x(\beta\beta')\varphi^{-1} = x(\alpha\epsilon_c) = c,$$

a contradiction.

For the rest, we can prove in the same fashion. \square

Lemma 3.10. *For each $x \in [k]$ and $a_1 < a_2 < a_3 < a_4 < a_5$ in X' , there is $t \in [t_k]$ such that $\begin{pmatrix} (\leftarrow [k]) & [k] \cap (\leftarrow x) & x & (x \rightarrow) \cap [k] & ([k] \rightarrow) \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix} \varphi$ is*

$$\begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & t & (t \rightarrow) \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4 & \bar{a}_5 \end{pmatrix} \text{ or } \begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & t & (t \rightarrow) \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_5 & \bar{a}_4 & \bar{a}_3 & \bar{a}_2 & \bar{a}_1 \end{pmatrix},$$

up to θ_φ .

Proof. Suppose that θ_φ is order-isomorphic. We set

$$\alpha_x = \begin{pmatrix} (\leftarrow [k]) & [k] \cap (\leftarrow x) & x & (x \rightarrow) \cap [k] & ([k] \rightarrow) \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}$$

and let $\mu = \begin{pmatrix} (\leftarrow a_2) & [a_2, a_4] & (a_4 \rightarrow) \\ a_1 & a_3 & a_5 \end{pmatrix}$. Then $\alpha_x\mu = \begin{pmatrix} (\leftarrow [k]) & [k] & ([k] \rightarrow) \\ a_1 & a_3 & a_5 \end{pmatrix}$. It is clear that $a_3(\alpha_x\mu)^{-1} = [k]$, by Lemma 3.9, $\bar{a}_3((\alpha_x\varphi)(\mu\varphi))^{-1} = [t_k]$. Since $\bar{a}_3(\mu\varphi)^{-1} \cap Y' = [\bar{a}_2, \bar{a}_4] \cap Y'$ and $\text{ran}(\alpha_x\varphi) = (\text{ran} \alpha_x)\theta_\varphi$. Then

$$\bar{a}_2(\alpha_x\varphi)^{-1} \cup \bar{a}_3(\alpha_x\varphi)^{-1} \cup \bar{a}_4(\alpha_x\varphi)^{-1} = [t_k].$$

By Lemma 3.7, $\bar{a}_3(\alpha_x\varphi)^{-1} = \{t\}$ for some $t \in [t_k]$. This shows that

$$\alpha_x\varphi = \begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & t & (t \rightarrow) \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4 & \bar{a}_5 \end{pmatrix}.$$

□

Immediate corollary of Lemma 3.10 is:

Corollary 3.11. *For each $x \in [k]$ and $a_1 < a_2 < a_3 < a_4 < a_5$ in X' , we have that $\begin{pmatrix} (\leftarrow [k]) & [k] \cap (\leftarrow x) & (x \rightarrow) \cap [k] & ([k] \rightarrow) \\ a_1 & a_2 & a_4 & a_5 \end{pmatrix}\varphi$ is*

$$\begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & (t \rightarrow) \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_4 & \bar{a}_5 \end{pmatrix} \text{ or } \begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & [t \rightarrow] \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_5 & \bar{a}_4 & \bar{a}_2 & \bar{a}_1 \end{pmatrix},$$

up to θ_φ , and $\begin{pmatrix} (\leftarrow [k]) & [k] \cap (\leftarrow x) & [x \rightarrow] \cap [k] & ([k] \rightarrow) \\ a_1 & a_2 & a_4 & a_5 \end{pmatrix}\varphi$ is

$$\begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & [t \rightarrow] \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_4 & \bar{a}_5 \end{pmatrix} \text{ or } \begin{pmatrix} (\leftarrow [t_k]) & [t_k] \cap (\leftarrow t) & (t \rightarrow) \cap [t_k] & ([t_k] \rightarrow) \\ \bar{a}_5 & \bar{a}_4 & \bar{a}_2 & \bar{a}_1 \end{pmatrix},$$

up to θ_φ .

Assume that $|X'| \geq 5$. Choose $a_1, a_2, a_3, a_4, a_5 \in X'$ such that $a_1 < a_2 < a_3 < a_4 < a_5$. For each $x \in [k]$, we let

$$\eta_x = \begin{pmatrix} (\leftarrow [k]) & [k] \cap (\leftarrow x) & (x \rightarrow) \cap [k] & ([k] \rightarrow) \\ a_1 & a_2 & a_4 & a_5 \end{pmatrix}$$

as an order-preserving transformation in $T_{OP}(X, X')$. It is clear that $a_2\eta_x^{-1} = [k] \cap (\leftarrow x)$. Define a bijection f from $[k]$ onto $[t_k]$ via

$$f(x) = \begin{cases} \max(\bar{a}_2(\eta_x\varphi)^{-1}) & \text{if } \theta_\varphi \text{ is order-isomorphic} \\ \min(\bar{a}_2(\eta_x\varphi)^{-1}) & \text{if } \theta_\varphi \text{ is order-anti-isomorphic.} \end{cases}$$

f is well-defined by Corollary 3.11. In next lemma, we will show that f is order-(anti)-isomorphic.

Lemma 3.12. *Suppose $|X'| \geq 5$ and θ_φ is order-(anti)-isomorphic. Then for each $[k] \in \{X \setminus X'\}$, $[k]$ and $[t_k]$ are order-(anti)-isomorphic.*

Proof. Suppose that θ_φ is order-isomorphic. Recall the bijective map $f : [k] \rightarrow [t_k]$ by sending $x \mapsto \max(\bar{a}_2(\eta_x\varphi)^{-1})$. It remains to show that f is order-preserving. Let $x_1, x_2 \in [k]$ be such that $x_1 < x_2$. Set

$$\beta = \begin{pmatrix} (\leftarrow [k]) & [k] \cap (\leftarrow x_1) & (x_1, x_2] & (x_2 \rightarrow) \cap [k] & ([k] \rightarrow) \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} (\leftarrow a_2) & [a_2, a_3] & [a_3, a_4] & (a_4, \rightarrow) \\ a_1 & a_2 & a_4 & a_5 \end{pmatrix} \text{ and } \beta_2 = \begin{pmatrix} (\leftarrow a_2) & [a_2, a_3] & (a_3, a_4] & (a_4, \rightarrow) \\ a_1 & a_2 & a_4 & a_5 \end{pmatrix}.$$

Then

$$\beta\beta_1 = \eta_{x_1} \quad \text{and} \quad \beta\beta_2 = \eta_{x_2}.$$

We have

$$\begin{aligned} f(x_1) &= \max(\bar{a}_2(\eta_{x_1}\varphi)^{-1}) = \max(\bar{a}_2((\beta\varphi)(\beta_1\varphi))^{-1}) \quad \text{and} \\ f(x_2) &= \max(\bar{a}_2(\eta_{x_2}\varphi)^{-1}) = \max(\bar{a}_2((\beta\varphi)(\beta_2\varphi))^{-1}). \end{aligned}$$

By Proposition 2.4 and Corollary 3.2, we obtain that

$$\begin{aligned} \bar{a}_2(\eta_{x_1}\varphi)^{-1} &= \bar{a}_2((\beta\varphi)(\beta_1\varphi))^{-1} = \bar{a}_2(\beta\varphi)^{-1} \quad \text{and} \\ \bar{a}_2(\eta_{x_2}\varphi)^{-1} &= \bar{a}_2((\beta\varphi)(\beta_2\varphi))^{-1} = \{\bar{a}_2, \bar{a}_3\}(\beta\varphi)^{-1}. \end{aligned}$$

Therefore $f(x_1) < f(x_2)$. This shows that f is order-isomorphic. \square

Then the following main result is proved.

Theorem 3.13. *Suppose that $|X'| \geq 5$. Then $T_{OP}(X, X') \cong T_{OP}(Y, Y')$ if and only if there is an order-(anti)-isomorphism f from X onto Y such that $(X')f = Y'$.*

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